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INVARIANT SETS AND SYMMETRIC PERIODIC MOTIONS OF REVERSIBLE MECHANICAL SYSTEMS†

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A method of constructing and classifying all symmetric periodic motions of a reversible mechanical system is proposed. The principal solution of the above problem is given for the Hill problem, the restricted three-body problem (including the photogravitational problem), the problem of a heavy rigid body with a fixed point, and that of a heavy rigid body on a rough plane. In particular, problems requiring a systematic numerical study are thereby formulated.

1. THE CONSTRUCTION OF SYMMETRIC PERIODIC MOTIONS OF REVERSIBLE SYSTEMS

Mechanical systems constitute [1] a class of linearly reversible systems of the form

$$\mathbf{u}^{\cdot} = \mathbf{U}(\mathbf{u}, \mathbf{v}), \quad \mathbf{v}^{\cdot} = \mathbf{V}(\mathbf{u}, \mathbf{v}); \quad \mathbf{u} \in \mathbf{R}^l, \quad \mathbf{v} \in \mathbf{R}^n \quad (l \geqslant n) \tag{1.1}$$

$$\mathbf{U}(\mathbf{u}, -\mathbf{v}) = -\mathbf{U}(\mathbf{u}, \mathbf{v}), \quad \mathbf{V}(\mathbf{u}, -\mathbf{v}) = \mathbf{V}(\mathbf{u}, \mathbf{v}) \tag{1.2}$$

Condition (1.2) means that system (1.1) is invariant under the mapping $(\mathbf{u}, \mathbf{v}) \to (\mathbf{u}, -\mathbf{v})$ if the time t is reversed simultaneously. The set $\mathbf{M} = \{\mathbf{u}, \mathbf{v}: \mathbf{v} = 0\}$ is invariant under this mapping.

Along with the solution $\mathbf{u} = \mathbf{u}(t)$, $\mathbf{v} = \mathbf{v}(t)$ system (1.1) also has the solution $\mathbf{u} = \mathbf{u}(-t)$, $\mathbf{v} = \mathbf{v}(-t)$ (Fig. 1a). The solutions are the same if $\mathbf{v}(0) = \mathbf{0}$ (Fig. 1b). In the case when $\mathbf{M}$ is intersected twice by the trajectory (Fig. 1b) we have a periodic motion that is symmetric about $\mathbf{M}$, which is defined by the Heinbockel–Struble theorem [2].

Here one can definitely observe a relation with the phase plane method for a conservative system with one degree of freedom, which is obviously a system of the form (1.1), (1.2). In the conservative system considered here all periodic motions are symmetrical about the abscissa axis.

Asymmetric periodic motion may also exist in the reversible system (1.1), (1.2) [3]. Moreover, the presence of domains of dissipative and conservative behaviour is "typical" for the system.

The simplest reversible system in the plane

$$u^{\cdot} = uv, \quad v^{\cdot} = u + \cos v \quad (-\pi < v \leqslant \pi)$$

has three singular points: (–1, 0), (0, $\pm\pi/2$). The first point belongs to the invariant set, the other two being symmetric about the u axis (Fig. 2). The singular point on the abscissa axis is stable in the Lyapunov sense and is surrounded by periodic solutions (centre), and the point in the upper half-plane is unstable, all the trajectories in its neighbourhood being outgoing trajectories. The singular point in the lower half-plane is symptotically stable. Another example of asymptotic stability (in some of the variables) in a mechanical system of the form (1.1), (1.2) is provided by the problem of the Celtic stone [4].

A solution $\mathbf{u} = \mathbf{u}(t)$, $\mathbf{v} = \mathbf{v}(t)$ of system (1.1), (1.2) is symmetric about $\mathbf{M}$ if the sets $\{\mathbf{u}(t), \mathbf{v}(t)\}$ and $\{\mathbf{u}(-t), -\mathbf{v}(-t)\}$ are the same. Thus, such a solution intersects the invariant set at time $t = 0$, and in the case of a T-periodic symmetric solution it also intersects $\mathbf{M}$ at time $t = T/2$.

Therefore, the Heinbockel–Struble theorem gives the necessary and sufficient conditions for the existence of a symmetric periodic solution. The assertion has a non-local nature. Local symmetric periodic motions form a Lyapunov family [5–7]. An extension of the Heinbockel–Struble theorem to the case of a torus is given in [8].

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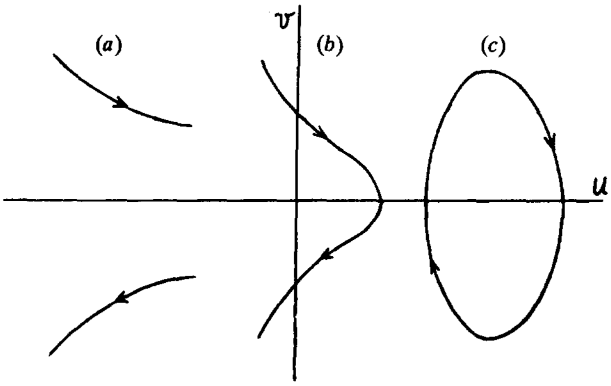

Fig. 1.

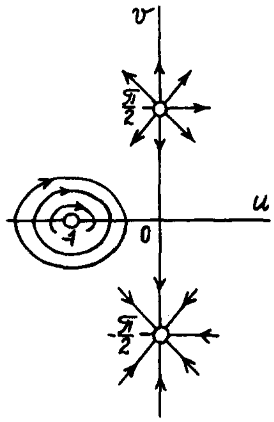

Fig. 2.

Let $\mathbf{M}^\tau$ be the image of the invariant set $\mathbf{M}$ at time τ obtained by means of (1.1). Then the intersection $\mathbf{M} \cap \mathbf{M}^\tau$ contains all the points of the invariant set that belong to symmetric periodic motions of period $2\tau/k$ ($k = 1, 2, 3, \ldots$). In this way, by varying τ one can construct all symmetric periodic motions of a specified system of the form (1.1), (1.2).

Theorem 1. Let Ω^τ be the set of all symmetric $(2r/k)$-periodic motions ($k = 1, 2, 3, \ldots$) of the reversible system (1.1), (1.2). Then $\Omega^\tau \cap \mathbf{M} \subset \mathbf{M} \cap \mathbf{M}^\tau$.

This assertion is fundamental. All initial points of symmetric periodic motions are contained in $\cup_{0<\tau<+\infty}(\mathbf{M} \cap \mathbf{M}^\tau)$. The problem of the construction and classification of all such motions can be solved in this way. The set $\cup_\tau(\mathbf{M} \cap \mathbf{M}^\tau)\backslash\cup_\tau(\Omega^\tau \cap \mathbf{M})$ consists of invariant manifolds belonging to an invariant set.

The set Ω^τ can be constructed numerically. Two problems arise: (a) the construction of $\mathbf{M}^\tau$, (b) the search for points belonging to $\mathbf{M} \cap \mathbf{M}^\tau$. The first problem can be solved in the standard way by numerical integration. To determine points in the intersection of $\mathbf{M}$ and $\mathbf{M}^\tau$, given that $\mathbf{M}^\tau$ is constructed approximately with some accuracy provided by the computer, is a complicated problem. For $n = 1$ the problem can be solved correctly using the Cauchy theorem on the mean value of a continuous function. For $n > 1$ a correct solution is possible, provided that the system admits of first integrals.

This approach is free from the disadvantage inherent in the numerical determination of invariant points of the corresponding mapping defined by a differential equation. If in the latter method a stationary point is defined to within some (possibly high) accuracy, there is no guarantee that the point corresponds to a periodic motion rather than to a torus. Therefore the problem of the classification of periodic motions becomes meaningless.

Numerous examples of the construction and classification of symmetric periodic motions (orbits) can be found in celestial mechanics [9–18]. Euler was the first [12] to construct such orbits (Lyapunov family) in the neighbourhood of one of the collinear libration points in the restricted three-body problem, which he introduced. Hill [13] gave an example of orbits that are symmetric relative to two invariant sets simultaneously. For the restricted three-body problem the Poincaré solutions of all three kinds are symmetric [14]. For the plane version of this problem Whittaker proposed [9] a criterion for the periodicity of an orbit, which is close to that stated in Theorem 1. Theorem 1 was in fact used in a numerical study at the Copenhagen observatory [9]. An approach similar to that considered above was proposed earlier in [15], but the condition used there is not sufficient for an orbit to be periodic.

A mechanical system can be invariant under several mappings $(t, \mathbf{x}) \to (-t, \mathbf{G}_j\mathbf{x})$ simultaneously, where $\mathbf{x} = (\mathbf{u}, \mathbf{v})^T$ is a vector in phase space (T denotes transposition) [1, 7]. Now, if $\mathbf{G}_j$ is not restricted by the condition $\mathbf{G}^2_j = id$ (the identity mapping), then we arrive at the system

$$\mathbf{x}^\cdot = \mathbf{X}(\mathbf{x}), \quad \mathbf{G}_j\mathbf{X}(\mathbf{x}) + \mathbf{X}(\mathbf{G}_j\mathbf{x}) \equiv \mathbf{0}, \quad \mathbf{x} \in \mathbf{R}^m \tag{1.3}$$

where $\mathbf{G}_j$ is a non-degenerate mapping.

It is obvious that system (1.3) is also reversible with the mapping

$$\mathbf{G}^s = \mathbf{G}_1^{s_1} \odot \cdots \odot \mathbf{G}_k^{s_k} \; (s_1 + \ldots + s_k = 2\alpha + 1; \;\; s_1, \ldots s_k, \;\; \alpha \in \mathbb{Z})$$

Thus, the invariant set has the form

$$\mathbf{M} = \bigcup_s \mathbf{M}_s, \quad \mathbf{M}_s = \{\mathbf{x}: \ \mathbf{x} = \mathbf{G}^s\mathbf{x}\} \tag{1.4}$$

Theorem 2. Theorem 1 remains true for system (1.3) with the invariant set $\mathbf{M} = \{\mathbf{u}, \mathbf{v}: \mathbf{v} = 0\}$ replaced by (1.4).

Let us now consider a reversible system (1.1) or (1.3) 2π-periodic in t and invariant under the mapping $(t, \mathbf{u}, \mathbf{v}) \to (-t, \mathbf{u}, -\mathbf{v})$ or $(t, \mathbf{x}) \to (-t, \mathbf{G}_j\mathbf{x})$. Then the following result holds.

Theorem 3. The set $\mathbf{M} \cap \mathbf{M}^\pi$ contains the initial points for all symmetric $(2\pi/k)$-periodic motions $(k = 1, 2, 3, \ldots)$.

2. THE HILL PROBLEM

A limiting version of the restricted three-body problem, that is, the Hill problem is of considerable interest because of its importance in the theory of the motion of the Moon or a natural satellite of any other planet. From the mathematical point of view it is a beautiful example of a simple non-integrable reversible system with two invariant sets. Fundamental studies of this problem are related to the proof of the existence and the construction of symmetric periodic orbits [13, 16, 17].

The equations of motion of the Moon with coordinates (x, y) will be considered in a frame of reference revolving at angular velocity m (the ratio of the mean motions of the Sun and the Moon around the Earth) with origin at the centre of the Earth and the abscissa axis directed towards the Sun. Then

$$\begin{gathered}\frac{d^2x}{d\tau^2} - 2m\frac{dy}{d\tau} + \frac{kx}{\rho^3} = 3m^2x \\ \frac{d^2y}{d\tau^2} + 2m\frac{dx}{d\tau} + \frac{ky}{\rho^3} = 0, \quad \rho^2 = x^2 + y^2\end{gathered} \tag{2.1}$$

where τ is the dimensionless time and k is a constant. We have $m = 0.08084893$ for the Moon and $m = -0.1461537$ for the eighth satellite of Jupiter.

System (2.1) admits of the energy integral

$$x'^2 + y'^2 = 3m^2x^2 + 2k/\rho + h \quad (h = \text{const}) \tag{2.2}$$

where the prime denotes differentiation with respect to τ.

System (2.1) is a linearly reversible system of the form (1.1), (1.2) with two invariant sets: $\mathbf{M}_1 = \{x, y, x', y': y = 0, x' = 0\}$ and $\mathbf{M}_2 = \{x, y, x', y': x = 0, y' = 0\}$. On $\mathbf{M}_1$ we have

$$y'^2 = 3m^2x^2 + 2k/|x| + h \tag{2.3}$$

On the other hand, if condition (2.3) is satisfied at some instant, then from (2.2) and (2.3) we obtain

$$x'^2 = 2k/\rho - 2k/|x| \leqslant 0$$

which is possible only if $x' = 0, y = 0$, that is, (2.3) is a necessary and sufficient condition for the inclusion in the fixed-point set $\mathbf{M}_1$.

By analogy, on $\mathbf{M}_2$ we have

$$x'^2 = 2k/|y| + h \tag{2.4}$$

By (2.2) we obtain

$$y'^2 = [3m^2 - k/|y|^3]x^2 + |y|^{-1}o(x^2/y^2)$$

It follows that $x = 0, y' = 0$ whenever $|y|^3 < k/(3m^2)$.

The following assertion has been proved.

Theorem 4. For the motion at a given energy level h, (2.3) is a necessary and sufficient condition for a point of the phase space to belong to the fixed-point set $\mathbf{M}_1$, and so is (2.4) for the set $\mathbf{M}_2$ in the domain $|y|^3 < k/(3m^2)$.

It follows that to construct all 2π-periodic motions symmetric about $\mathbf{M}_1$ or $\mathbf{M}_2$ at a fixed energy level it is necessary to consider the set (2.3) or (2.4) for $|y|^3 < k/(3m^2)$ (respectively, the sets $\mathbf{M}_{1h}$ and $\mathbf{M}_{2h}$), to construct $\mathbf{M}^{\tau}_{1h}$ ($\mathbf{M}^{\tau}_{2h}$), and to determine the points belonging to the intersection $\mathbf{M}_{1h} \cap \mathbf{M}^{\tau}_{1h}$ ($\mathbf{M}_{2h} \cap \mathbf{M}^{\tau}_{2h}$). Since $\mathbf{M}_{1h}$ and $\mathbf{M}_{2h}$ consist of curves, the problem of finding the points of intersection also has a correct solution in the case when $\mathbf{M}^{\tau}_{1h}$ and $\mathbf{M}^{\tau}_{2h}$ are constructed numerically with given accuracy.

Let $h = -\nu < 0$. Then the Hill method yields the domain

$$3m^2x^2 + 2k(x^2+y^2)^{-1/2} \geqslant \nu \tag{2.5}$$

of possible motions in the (x, y) plane (the unhatched area in Fig. 3). It follows that by considering (2.4) as a condition for the inclusion in $\mathbf{M}_2$ we can construct all periodic motions symmetric about $\mathbf{M}_2$ for $\nu > \nu^* = 3(3m^2k^2)^{1/2}$ because there are no such motions outside the unhatched oval. Furthermore, if a motion starts in one of the domains 1, 2, or 3, it will remain there forever.

In the case $\nu > \nu^*$ the set (2.3) is shown in Fig. 4. Equations (2.1) have two constant solutions $x = \pm(k/(3m^2))^{1/2}$, $y = 0$ corresponding to relative equilibrium positions. The characteristic equation formed for these solutions has two real roots $\pm(1+\sqrt{28})^{1/2}m$ and a pair of purely imaginary roots $\pm(1-\sqrt{28})^{1/2}m$. Since the relative equilibria belong to the invariant set $\mathbf{M}_1$, a Lyapunov family of symmetric periodic orbits corresponds to a pair of pure imaginary roots. Thus, for ν close to ν^* transitions of form 1 exist (Fig. 4). Whether such transitions are maintained or not as ν^* increases can be decided as a result of numerical investigations.

We shall now consider small values of m. When $m = 0$ we have the two-body problem and the system admits of the solution

$$x = \rho(\theta)\cos\theta, \quad y = \rho(\theta)\sin\theta \tag{2.6}$$

$$\rho(\theta) = \frac{a(1-e^2)}{1+e\cos\theta}, \quad \rho^2(\theta)\frac{d\theta}{d\tau} = [ka(1-e^2)]^{1/2}, \quad a = \frac{k}{\nu}$$

where e is the eccentricity, the motion being elliptic when $|e| < 1$ and, in particular, circular when $e = 0$. For small $m \neq 0$ the solution in a finite time interval can be constructed as a series in m. Then, following [18], we obtain

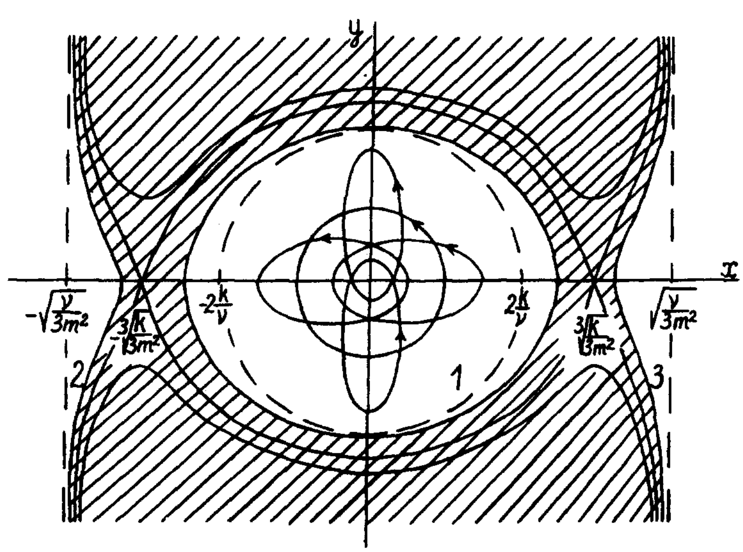

Fig. 3.

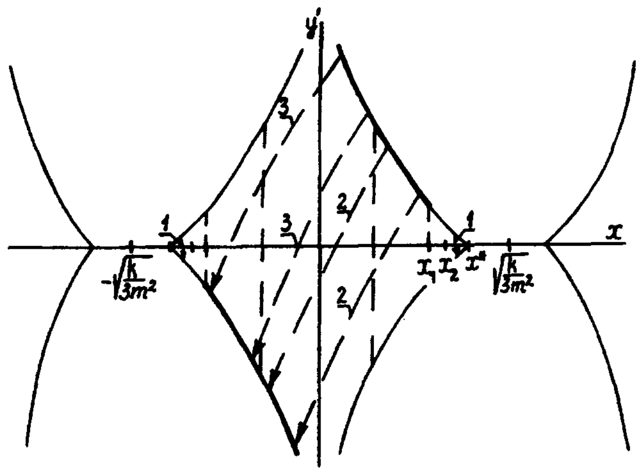

Fig. 4.

$$x = \rho(\theta)\{(1-\xi)\cos\theta + \eta\sin\theta\}, \quad y = \rho(\theta)\{(1-\xi)\sin\theta - \eta\cos\theta\} \tag{2.7}$$

$$\xi = m\{c_1[f(\theta)e^{\varkappa\theta} + f(-\theta)e^{-\varkappa\theta}] + c_2[f(\theta)e^{\varkappa\theta} - f(-\theta)e^{\varkappa\theta}]\} + O(m^2)$$

$$\eta = \int_0^\theta \{-2\xi(\theta) + m[\rho^2(\theta)(ka(1-e^2))^{-1/2} + c_3]d\theta + mc_4\} + O(m^2)$$

where c_s are constants, $f(\theta)$ is a certain 2π-periodic function of θ, and $\pm\varkappa$ are the characteristic exponents of the equation

$$\frac{d^2\xi}{d\theta^2} + \left(4 - \frac{3}{1+e\cos\theta}\right)\xi = 0$$

From the condition that this solution belongs to the set $\mathbf{M}_1$ for $\theta = 0$ we obtain $c_1 = c_4 = 0$. The solution (2.7) also belongs to $\mathbf{M}_1$ when $\theta = \pi$ if

$$c_2 g(\pi)(e^{\varkappa\pi} - e^{-\varkappa\pi}) + O(m) = 0, \quad g(\theta) = f'(\theta) + \varkappa f(\theta)$$

$$\int_0^\pi \{-2c_2[f(\theta)e^{\varkappa\pi} - f(-\theta)e^{-\varkappa\pi}] + \rho^2(\theta)(ka(1-e^2))^{-1/2} + c_3\}d\theta + O(m) = 0$$

Since for $0 < |e| < 1$ we have [18] $g(\pi) \neq 0$, $\varkappa \neq is$ ($s \in \mathbb{Z}$), for each of these values of e there is a unique system (c_1, c_2) that guarantees transitions of the form 2 ($-1 < e < 0$) and 3 ($0 < e < 1$) (Fig. 4). This means that for any sufficiently small m the whole specified curve in the first quadrant turns into the specified curve in the third quadrant if the mapping is constructed for $\tau = \pi(a^3/k)^{1/2}$, which corresponds to varying θ from 0 to π. It follows that the symmetric periodic orbits (2.6) can be extended with respect to m for all $|e| < 1$, including the case $e = 0$. For small $m \neq 0$ motion takes place inside the oval (Fig. 3).

Note that the presence of an invariant set $\mathbf{M}_2$ also implies the existence of orbits close to elliptic ones and symmetric about y for small $m \neq 0$ (Fig. 3).

For any given ν to each value x there correspond two values y' completely defined on $\mathbf{M}_1$. For ν close to ν^* and m small enough there are local periodic orbits ($x_2 < x < x^*$) and orbits close to elliptic ones ($x < x_1 = 2a + O(m)$). For $x_1 < x < x_2$ the question of the existence of symmetric periodic orbits remains open for numerical investigation. The determination of the boundary value of m such that all elliptic orbits remain generating is another problem, and so is the study of symmetric periodic orbits for transcritical values of m.

To study the question of the existence of periodic orbits symmetric about $\mathbf{M}_1$ and $\mathbf{M}_2$, we transform [16] system (2.1) to the form

$$\frac{d^2x}{d\tau^2}-2m\frac{dy}{d\tau}-\frac{3}{2}m^2x+\frac{kx}{\rho^3}=\frac{3}{2}\lambda x$$
$$\frac{d^2y}{d\tau^2}+2m\frac{dx}{d\tau}-\frac{3}{2}m^2y+\frac{ky}{\rho^3}=-\frac{3}{2}\lambda y \tag{2.8}$$

with Lyapunov parameter $\lambda = m^2$. Then for $\lambda = 0$ system (2.8) admits of the partial solution

$$x=a\cos\omega\tau,\ \ y=a\sin\omega\tau,\ \ k/a^3=\omega^2+2m\omega+1{,}5m^2 \tag{2.9}$$

which describes circular orbits of radius a For the given value of a the motion in one direction ($\omega/m >$ 0) evolves at lower angular velocity than the motion in the opposite direction. Thus, ω/m –3 corresponds to $\omega/m = 0$ and $\omega/m = -1$ corresponds to the maximum value $a = (2k/m^2)^{1/3}$.

The solution of system (2.8), identical with (2.9) for $\lambda = 0$, has the form

$$x=a\{(1-\lambda\xi)\cos\omega\tau+\lambda\eta\sin\omega\tau\},\ \ y=a\{(1-\lambda\xi)\sin\omega\tau-\lambda\eta\cos\omega\tau\} \tag{2.10}$$

where

$$\frac{d^2\xi}{d\tau^2}=-\varkappa^2\,\xi+2(\omega+m)\alpha\ldots,\ \ \frac{d\eta}{d\tau}=-2(\omega+m)\xi+\alpha+\ldots,\ \ \varkappa^2=\omega^2+2m\omega-\frac{m^2}{2}+\ldots$$

in the zeroth approximation with respect to λ (α is an arbitrary constant). The solution (2.10) at $\tau = 0$ belongs to $\mathbf{M}_1$ if $\eta(0) = 0$, $\xi'(0) = 0$, and for $\tau = \pi/(2|\omega|)$ it intersects $\mathbf{M}_2$ if η and ξ are also equal to zero at that time. These conditions are satisfied by a unique value $\xi(0)$ at $\alpha = 0$ and $\varkappa^2 \neq N^2\omega^2$ ($N = 0, 1, 2, \ldots$).

It follows that if h is given, there is a unique point on $\mathbf{M}_1$ such that the trajectory starting from this point at $\tau = \pi/(2|\omega|)$ reaches the fixed-point set $\mathbf{M}_2$. It follows that for this choice of τ the image of $\mathbf{M}^\tau{}_1$ intersects $\mathbf{M}_2$ at a single point if one of the branches of $\mathbf{M}_1$ from one quadrant is considered.

Periodic orbits symmetric simultaneously with respect to $\mathbf{M}_1$ and $\mathbf{M}_2$ can have radii $a \leqslant [2k/(3m^2)]^{1/3}$. For $a < [k/(3m^2)]^{1/3}$ all such orbits can be constructed using Theorem 4. For larger values of a a direct verification of the conditions $x = 0, y' = 0$ on $\mathbf{M}_2$ is necessary.

3. THE THREE-BODY PROBLEM

In the restricted three-body problem one studies the orbits of a point **P** of negligible mass in the gravitational field of two bodies (point masses) **S** and **J**. It is assumed that **P** has no effect on the motion of **S** or **J**. For a suitable choice of the scale and constants, the equations of motion for **P** have the form [9]

$$x^{\cdot\cdot}-2y^{\cdot}=\partial U/\partial x,\ \ y^{\cdot\cdot}+2x^{\cdot}=\partial U/\partial y \tag{3.1}$$

$$U=(1-\mu)/R_1+\mu/R_2+(x^2+y^2)/2 \tag{3.2}$$
$$R_1^2=(x+\mu)^2+y^2,\ \ R_2^2=(x+\mu-1)^2+y^2$$

for the plane version of the problem, where $1-\mu$ and μ are the dimensionless masses of **S** and **J**, and R_1 and R_2 are the distances from **P** to **S** and **J**, respectively.

Below we consider a more general problem when **P** is also subject to radiation pressure from the radiation of one or two of the bodies **S** and **J**. In this case we have the photogravitational three-body problem, in which the force function U has the form [19]

$$U=Q_1(1-\mu)/R_1+Q_2\mu/R_2+(x^2+y^2)/2 \tag{3.3}$$

where the physically admissible parameters Q_1 and Q_2 belong to the interval $(-\infty, 1]$. When $Q_1 = Q_2 = 1$, we have the classical problem (3.1), (3.2).

System (3.1) with potential (3.2) or, more generally, with potential (3.3) is a linearly reversible system of the form (1.1), (1.2) with an invariant set $\mathbf{M} = \{x, y, x', y' : y = 0, x' = 0\}$, where $l = n = 2$. However, the presence of the energy integral

$$x'^2 + y'^2 = 2Q_1(1-\mu)/R_1 + 2Q_2\mu/R_2 + (x^2+y^2) + h \quad (h = \text{const}) \tag{3.4}$$

enables us to apply correctly the method in Section 1 to construct and classify symmetric periodic orbits.

In the fixed-point set $\mathbf{M}$ we obtain

$$y'^2 = 2Q_1(1-\mu)/|x+\mu| + 2Q_2\mu/|x+\mu-1| + x^2 + h \tag{3.5}$$

If this equality is satisfied, we obtain from (3.4)

$$x'^2 = [1 - Q_1(1-\mu)/|x+\mu|^3 - Q_2\mu/|x+\mu-1|^3]y^2 + o(y^2)$$

The assertion below follows.

Theorem 5. If $|y|$ is sufficiently small compared to $|x + \mu|$ and $|x + \mu + 1|$, then (3.5) is an necessary and sufficient condition for belonging to the invariant set in the part of the phase space where

$$g(x) = Q_1(1-\mu)/|x+\mu|^3 + Q_2\mu/|x+\mu-1|^3 > 1 \tag{3.6}$$

Obviously, (3.6) is not satisfied for any x if the force due to light pressure exceeds the gravitational force for each of the main bodies ($Q_1 < 0$, $Q_2 < 0$), and also in the degenerate case $Q_1 = Q_2 = 0$.

We consider the case when $Q_1 \geqslant 0$, $Q_2 \geqslant 0$, where $Q_1 + Q_2 > 0$, which includes the case of the classical three-body problem $Q_1 + Q_2 = 1$. The graph of $g(x)$ is shown in Fig. 5. One can see that the minimum value is reached at $x^* = 1/(1 + \alpha) - \mu$ and

$$g(x^*) = (1+\alpha)^3[Q_1(1-\mu) + Q_2\mu/\alpha^3] \qquad \alpha = [Q_2\mu/(Q_1(1-\mu))]^{1/4}$$

If $g(x^*) > 1$, then Theorem 5 holds in the domain where $x_1 < x < x_2$. Otherwise the strip $x_3 \leqslant x \leqslant x_4$ is beyond the scope of our consideration.

The domain of possible motions can be determined from (3.4). It follows that for $\nu > \nu^*$ ($\nu = -h$), where ν^* is a positive number, a bounded invariant domain exists in the (x, y) plane for which all symmetric periodic orbits can be constructed on the basis of Theorem 5. To do this it is necessary to consider manifold (3.5) in domain (3.6) for a fixed constant energy value, and to construct the image of $\mathbf{M}^\tau$ and define the points in the intersection $\mathbf{M} \cap \mathbf{M}^\tau$ belonging to (3.6). The last problem can be

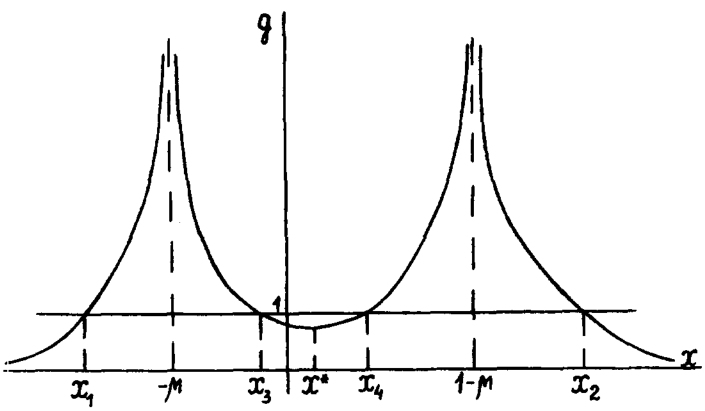

Fig. 5.

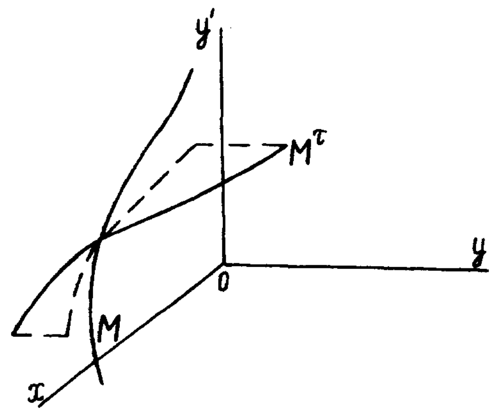

Fig. 6.

solved correctly also for a numerical construction of $\mathbf{M}^\tau$ because it is guaranteed that $|y|$ is small in this case (Fig. 6).

Note that in the Copenhagen version [9] of the three-body problem ($Q_1 = Q_2 = 1, \mu = 1/2$) we have $\alpha = 1, x^* = 0, g(x^*) = 8$ and (3.6) is satisfied. Thus, symmetric periodic orbits can be constructed in the domain where $x_1 < -0.5$ and $x_2 < 0.5$.

4. A HEAVY RIGID BODY WITH A FIXED POINT

Retaining the standard notation of [20], we write the equations of motion of the problem

$$Adp/dt + (C-B)qr = P(\gamma_2 z_c - \gamma_3 y_c), \quad d\gamma_1/dt = r\gamma_2 - q\gamma_3 \tag{4.1}$$

$$(pqr, ABC, xyz, 123)$$

System (4.1) admits of three integrals, namely, the energy, the kinetic momentum, and the geometric integral

$$\begin{gathered} Ap^2 + Bq^2 + Cr^2 + 2P(x_c\gamma_1 + y_c\gamma_2 + z_c\gamma_3) = 2h \quad (h = \text{const}) \\ Ap\gamma_1 + Bq\gamma_2 + Cr\gamma_3 = \beta \quad (\beta = \text{const}), \quad \gamma_1^2 + \gamma_2^2 + \gamma_3^2 = 1 \end{gathered} \tag{4.2}$$

Let $y_c = 0$. Then, apart from the invariant set $\{\gamma_1, \gamma_2, \gamma_3, p, q, r: p = q = r = 0\}$ system (4.1) has the invariant set $\mathbf{M} = \{\gamma_1, \gamma_2, \gamma_3, p, q, r: \gamma_2 = 0, q = 0\}$. We obtain a system of the form (1.1) with $l = 4$ and $n = 2$.

Periodic motions symmetric with respect to $\mathbf{M}$ are of considerable interest. They include the Grioli regular precessions

$$p = n(x_c - z_c\cos\tau)/l, \quad q = n\sin\tau, \quad r = n(z_c + x_c\cos\tau)/l, \quad \tau = nt - \varepsilon + \pi/2$$

$$\gamma_1 = -\frac{n^2}{Pl^2}[Cz_c\cos\tau + (B-C)x_c\sin^2\tau]$$

$$\gamma_2 = \frac{n^2}{Pl^3}[(Ax_c^2 + Cz_c^2)\sin\tau - (A-C)x_c z_c \sin\tau\cos\tau]$$

$$\gamma_3 = \frac{n^2}{Pl^2}[Ax_c\cos\tau + (A-B)z_c\sin^2\tau]$$

$$l^2 = x_c^2 + z_c^2, \quad n^4 = \frac{P^2 l^2}{(A-B)(B-C) + (A-B+C)^2}$$

(ε is a constant), which are possible [21] when $x_c\sqrt{(B-C)} = z_c\sqrt{(A-B)}(A > B > C)$. For these motions the axis of precession is not vertical, the angle between the axis and the vertical direction being δ, where

$$\cos\delta = n^2(A-B-C)/(Pl)$$

On the fixed-point set $\mathbf{M}$ we have

$$\gamma_1^2+\gamma_3^2=1,\quad Ap\gamma_1+Cr\gamma_3=\beta,\quad Ap^2+Cr^2+2P(x_c\gamma_1+z_c\gamma_3)=2h \tag{4.3}$$

On the other hand, when (4.3) holds, we deduce that $\gamma_2 = 0$ and $q = 0$ from integrals (4.2).

Theorem 6. Conditions (4.3) are necessary and sufficient for belonging to the fixed-point set $\mathbf{M}$.

Thus, to construct all symmetric periodic motions for fixed h and β we define the curves $\Gamma_* = \{\gamma_1 = \cos\alpha, \gamma_2 = 0, \gamma_3 = \sin\alpha, p = p(\alpha), q = 0, r = r(\alpha)\}$ with $\Gamma_* \subset \mathbf{M}$, we then construct their projections $\Gamma_p = \{\alpha, p{:}\, p = (\alpha)\}$ ($\Gamma_r = \{\alpha, r{:}\, r = r(\alpha)\}$) onto the (p, α) or (r, α) planes, and we find the image Γ_*^τ and its projection Γ_p^τ or Γ_r^τ. Then a symmetric $(2\pi/k)$-periodic ($k \in \mathbb{N}$) motion corresponds to the value α^* at the point of intersection of Γ_p and Γ_p^τ (Γ_p and Γ_p^τ) (Fig. 7).

The existence of α^* can also be shown in the numerical construction of Γ_*^τ, because to every point in $\Gamma_p \cap \Gamma_p^\tau$ ($\Gamma_r \cap \Gamma_r^\tau$) there corresponds a point in $\Gamma_* \cap \Gamma_*^\tau$, and the other way round. The problem of the intersection of $\Gamma_{p,r}$ and $\Gamma_{p,r}^\tau$ can be solved correctly by the Cauchy theorem on the mean value of a continuous function.

5. A HEAVY RIGID BODY ON AN ABSOLUTELY ROUGH PLANE

The problem can be described by a closed system of six equations of the first order for the projections of ω and γ (or $\mathbf{r}$) [3]

$$\boldsymbol{\theta}\boldsymbol{\omega}^{\cdot}+\boldsymbol{\omega}\times(\boldsymbol{\theta}\cdot\boldsymbol{\omega})=mg\mathbf{r}\times\boldsymbol{\gamma}-m\mathbf{r}\times[\boldsymbol{\omega}^{\cdot}\times\mathbf{r}+\boldsymbol{\omega}\times\mathbf{r}^{\cdot}+\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r})],\quad \boldsymbol{\gamma}^{\cdot}+\boldsymbol{\omega}\times\boldsymbol{\gamma}=\mathbf{0} \tag{5.1}$$

Here m is the mass of the body, $\boldsymbol{\omega} = (\omega_1, \omega_2, \omega_3)^T$ is the vector of the instantaneous angular velocity of the body, g is the acceleration due to gravity, θ is the central tensor of inertia of the body, $\mathbf{r} = (x, y, z)^T$ is the radius vector of the point of contact of the body and the plane, and $\boldsymbol{\gamma} = (\gamma_1, \gamma_2, \gamma_3)^T$ is the unit vector of the vertical direction at this point pointing upwards.

A relation between $\mathbf{r}$ and $\boldsymbol{\gamma}$ can be established using the equation of the body surface. If the equation is written as $f(\mathbf{r}) = 0$, then

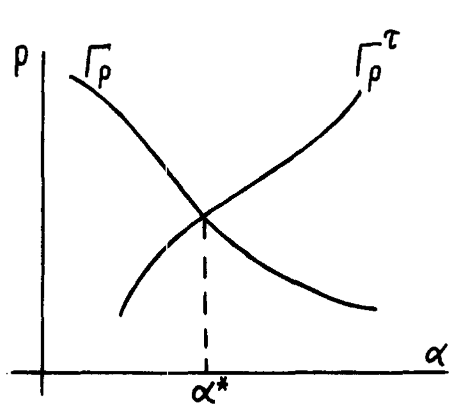

Fig. 7.

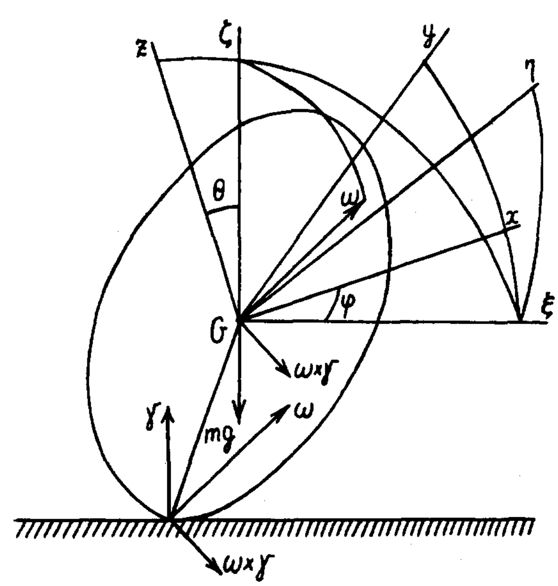

Fig. 8.

$$\gamma = -\operatorname{grad} f(\mathbf{r})/|\operatorname{grad} f(\mathbf{r})| \tag{5.2}$$

Below we consider the case when the axes of the attached system of coordinates are directed along the principal central axes of inertia and the body moves without jumps. In this case $\mathbf{R}\gamma > 0$, where

$$\mathbf{R} = mg\gamma - m[\boldsymbol{\omega}^{\cdot} \times \mathbf{r} + \boldsymbol{\omega} \times \mathbf{r}^{\cdot} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})]$$

is the reaction of the supporting plane.

System (5.1), (5.2) is linearly reversible, having the invariant set $\mathbf{M}_* = \{\gamma, \boldsymbol{\omega}: \boldsymbol{\omega} = \mathbf{0}\}$.

The presence of the first two integrals

$$m(\boldsymbol{\omega} \times \mathbf{r})^2 + \boldsymbol{\omega} \cdot (\boldsymbol{\theta} \cdot \boldsymbol{\omega}) - 2mg(\mathbf{r} \cdot \gamma) = 2h \ \ (h = \text{const}), \ \ \gamma_1^2 + \gamma_2^2 + \gamma_3^2 = 1$$

that is, the energy and the geometric integral, enables us to describe the problem by a system of four equations depending on h or a third-order non-autonomous system.

Making the substitution

$$\gamma_1 = \sin\theta\cos\varphi, \ \ \gamma_2 = \sin\theta\sin\varphi, \ \ \gamma_3 = \cos\theta \ \ (0 \leqslant \theta \leqslant \pi) \tag{5.3}$$
$$\omega_1 = p\cos\varphi - q\sin\varphi, \ \ \omega_2 = p\sin\varphi + q\cos\varphi, \ \ \omega_3 = r$$

we obtain

$$\theta^{\cdot} = -q, \ \ \varphi^{\cdot} = -r + p\operatorname{ctg}\theta \ \ (\sin\theta \neq 0)$$

Let $\chi = -r\sin\theta + p\cos\theta$ be the projection of $\boldsymbol{\omega} \times \gamma$ onto the $G\eta$ axis perpendicular to $G\xi$ in the coordinate plane xy, where G is the centre of mass of the body and $G\xi$ is the line of intersection of the xy plane with that passing through the Gz axis and the vertical direction (Fig. 8). Therefore $\chi = 0$ if $\boldsymbol{\omega} \times \gamma = \mathbf{0}$ or if $\boldsymbol{\omega} \times \gamma \neq \mathbf{0}$, but $\boldsymbol{\omega} \times \gamma$ belongs to the $G\xi z$ plane. When $\boldsymbol{\omega} \times \gamma = \mathbf{0}$, we have either $\boldsymbol{\omega} = \mathbf{0}$ or ω and γ are collinear vectors. If $\boldsymbol{\omega} = \mathbf{0}$ all the time, we have an equilibrium state. If $\boldsymbol{\omega} = \mathbf{0}$ for at least two instants of time, we obtain a periodic motion, since the set $\boldsymbol{\omega} = \mathbf{0}$ is contained in the invariant set $\mathbf{M}^*$. If ω becomes zero only once along the trajectory at $t = t^*$, then for $t > t^*$ ($t < t^*$) we have the general case of $\boldsymbol{\omega} \neq \mathbf{0}$. But if $\boldsymbol{\omega} \times \gamma = \mathbf{0}$, while $\boldsymbol{\omega} \neq \mathbf{0}$, then in the case of a point of contact with $\mathbf{r} \times \mathbf{R} = \mathbf{0}$ we have permanent rotation about the vertical axis.

Let $\boldsymbol{\omega} \times \gamma$ be a vector in the $G\xi z$ plane. Then the vector of the instantaneous angular velocity must be perpendicular to this plane. Such a position of ω at all times during the motion involves rolling in one direction on the $G\xi z$ plane. In particular $\varphi = 0$ or $\varphi = \pi/2$, and we have rolling on the main planes Gxy or Gyz, under the obvious condition that such motions are admissible.

The feasibility of these or other motions is determined by the positions of $\boldsymbol{\omega}$ and γ, and also by the form of the body surface. If the form is such that to the horizontal position $\boldsymbol{\omega}$ there corresponds a whole semi-trajectory of the system of equations (5.1) and (5.2), then all the remaining motions, with the exception of the equilibria and permanent rotations about the vertical axis, can be described by introducing new "time" φ.

We now consider the formulae for transformation (5.3). In these formulae the angles φ and θ and the new projections p, q and r have different parts to play. Below, this will manifest itself in that $\chi \neq 0$ if ω is a horizontal vector belonging at the same time to the $G\xi z$ plane. Otherwise, rolling such that one of the main planes is parallel to the vertical plane can also be described with the new "time" φ.

Lemma. The motion of a rigid body on an absolutely rough plane can be described with the new "time" φ. The only exceptions are the equilibrium states ($\boldsymbol{\omega} = 0$) and the permanent rotations about the vertical axis ($\boldsymbol{\omega} \times \gamma = 0$).

Note that the excluded motions are quite well known (see, for example, [4]).

As a result of changing to the new independent variable φ, we obtain the following 2π-periodic third-order system

$$dp / d\varphi = q + (\varphi^{\cdot} S)^{-1}\{(X + mxzr^{\cdot})[(B + m(x^2 + z^2))\cos\varphi + mxy\sin\varphi] +$$
$$+(Y + myzr^{\cdot})[(A + m(y^2 + z^2))\sin\varphi + mxy\cos\varphi]\}$$

$$dq/d\varphi = -p - (\varphi^{\cdot} S)^{-1}\{(X + mxzr^{\cdot})[(B + m(x^2+z^2))\sin\varphi - mxy\cos\varphi] + \tag{5.4}$$
$$+(Y + myzr^{\cdot})[-(A + m(y^2+z^2))\cos\varphi + mxy\sin\varphi]\}$$

$$d\theta/d\varphi = -q/\varphi^{\cdot}$$

$$X = (B-C)\omega_2\omega_3 + m\{g(\gamma_3 y - \gamma_2 z) - \omega_1(xx^{\cdot} + yy^{\cdot} + zz^{\cdot}) + x^{\cdot}(\omega_1 x + \omega_2 y + \omega_3 z) -$$
$$-\omega_3 y(\omega_1 x + \omega_2 y) + \omega_2^{\cdot} z(\omega_3 z + \omega_1 x) + yz(\omega_2^2 - \omega_3^2)\}$$
$$Y = (C-A)\omega_3\omega_1 + m\{g(\gamma_1 z - \gamma_3 x) - \omega_2(xx^{\cdot} + yy^{\cdot} + zz^{\cdot}) + y^{\cdot}(\omega_1 x + \omega_2 y + \omega_3 z) -$$
$$-\omega_1 z(\omega_2 y + \omega_3 z) + \omega_3 x(\omega_1 x + \omega_2 y) + zx(\omega_3^2 - \omega_1^2)$$
$$S = AB + Am(x^2+z^2) + Bm(y^2+z^2) + m^2 z^2(x^2+y^2+z^2), \quad \theta = \operatorname{diag}\{A, B, C\}$$

On the right-hand side of this system it is necessary to express $\mathbf{r}$ in terms of $\boldsymbol{\gamma}$ and to change to the new variables by (5.3). The projection r and velocity $r^{\cdot}$ are eliminated by means of the energy integral.

Suppose that the derivative f_y' is an odd function. Then system (5.1), (5.2) has one more fixed-point set $\mathbf{M} = \{\gamma_1, \gamma_2, \gamma_3, \omega_1, \omega_2, \omega_3: \gamma_2 = 0, \omega_2 = 0\}$. As a result, system (5.4) is invariant under the substitution $(\varphi, p, q, \theta) \to (-\varphi, p, -q, \theta)$, that is, it is a 2π-periodic linearly reversible system with the fixed-point set $\mathbf{M}_h = \{p, q, \theta: q = 0\}$. For such a system Theorem 3 enables us to construct all symmetric motions that are $(2\pi/k)$-periodic in φ ($k \in \mathbb{N}$). Clearly, by substituting $\varphi(t)$ in place of φ, where

$$t = \int_0^{\varphi} \frac{d\varphi}{-r(\varphi) + \operatorname{ctg}\theta(\varphi)} \tag{5.5}$$

we can construct all symmetric periodic motions of system (5.1), (5.2).

Theorem 7. Let $\mathbf{\Omega}^{\pi}{}_h$ be the set of all symmetric motions of system (5.4) that are $(2\pi/k)$-periodic in θ ($k \in \mathbb{N}$) for a given value of the constant energy h. Then $\mathbf{\Omega}_h^{\pi} \cap \mathbf{M}_h \subset \mathbf{M}_h \cap \mathbf{\Omega}_h^{\pi}$, and the initial values for all symmetric periodic motions of system (5.1), (5.2) with $f_y'(x, -y, z) = -f_y'(x, y, z)$ at a given level h can be determined from the formulae

$$\gamma_1 = \sin\theta_*, \quad \gamma_2 = 0, \quad \gamma_3 = \cos\theta_*, \quad \omega_1 = p_*, \quad \omega_2 = 0, \quad \omega_3 = r_*$$

where $(p_*, q_*, \theta_*) \in \mathbf{M}_h \cap \mathbf{M}_h^{\pi}$.

The problem of determining the points $\mathbf{M}_h \cap \mathbf{M}_h^{\pi}$ has a correct solution. Indeed, we can choose a straight line $\mathbf{\Gamma}_h = \{p, q, \theta: q = 0, \theta = \theta_0\}$, where θ_0 is a fixed number from the interval $(0, \pi]$, and construct $\mathbf{\Gamma}_h^{\pi} \subset \mathbf{M}_h^{\pi}$. Then $\mathbf{\Gamma}_h^{\pi}$ contains a point from $\mathbf{M}_h$ if q changes its sign on $\mathbf{\Gamma}_h^{\pi}$.

Finally, we observe that the problems in Sections 2–4 can be reduced to the investigation of a 2π-periodic reversible system and studied in the same way as the problem in Section 5.

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